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THE WORM CONJUGATED WITH AN ORDERED CURL OF INVOLUTE CYLINDRICAL SURFACES

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ABSTRACT

The involute teeth of straight or helical teethed wheels are usually machined with hob mill. The hob mills admit as primary peripheral surface a cylindrical helical surface with constant pitch. In this paper, based on the complementary theorem of "generating trajectories family", it is demonstrated that the worm conjugated with an ordered curl of involute flanks is also an involute worm.

The proposed development is based on analytical representations of the enwrapping surfaces with single point contact. In problem solving was applied the method of intermediary surface (generating rack gear), developing the issue as a succession of enwrapping surfaces with linear contact. As application, it is presented a solution of the same issue in a graphical design environment — CATIA.

Keywords: graphical method, CATIA, generating trajectories family

1. INTRODUCTION

The involute teethed wheels are usually machined by enwrapping, using the rolling method, with rack gear tools, gear shaped tools of hob mills [1, 2].

The constructive profiling of the worm cutter, assumes to know the primary form of the generating worm as base of active surface which represent the teeth of cutting tool — the hob mill.

The hob mill profiling can be made based on the fundamental theorems of surfaces enwrapping, the Olivier or Gohman theorem [1]. The second theorem of Olivier is particularly used, as theorem for enwrapping of surfaces with point contact, with intermediary surface's method. The intermediary surface is represented by the flanks of rack gear conjugated with a wheel with straight or helical teeth.

For the involute profile of the surfaces curl, at contact with a hob mill, were imagined solutions for determining the theoretical form of the worm, by Euler himself and, further by Kutzbach (1925). These solutions were given by Olivier fundamental theorems.

For the same problem type, solutions were proposed by Oancea [4], based on the Gohman theorem, using the method of intermediary surface.

They are methods known for generating various worms' types [2, 3, 5] such as the Archimedes's worm and involute worm with tools which materialize a straight line generatrix or for the unfolded worm, with an in-plane surface (grinding wheel) modelling a plane tangent to a helical surface.

Also, Litvin et al. [3] recommend techniques and solutions for generation of worms on type *K* and *F*.

For profiling the primary peripheral surface of the worm conjugated with involute teeth were imagined solutions and were created specialized algorithms in *LISP* for *AutoCAD* [6].

Also, were made applications in CATIA regarding the generation of helical surfaces [7] using the capabilities of this graphical design environment [8].

In this paper, is proposed an analytical approach for determining the worm conjugated with an ordered curl of involute cylindrical surfaces (involute straight teethed wheel), based on a complementary theorem of surface enwrapping — the family of generating trajectories [9]. The goal is to demonstrate, in an analytical way, that the worm conjugated with involute teeth is an involute worm.

2. INVOLUTE FLANK OF TEETHED WHEEL

In figure 1, is presented the frontal profile of the involute tooth, associated with a reference system, *XYZ*, where the *Z* axis is overlapped by the teethed wheel's axis. At the same time, is defined the $\xi\eta\zeta$ reference system associated with the rack gear's centrode and the *xyz* global reference system, initially overlapped by *XYZ* reference system.

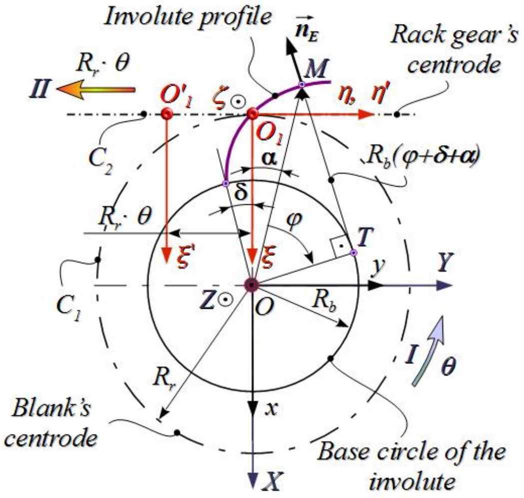

Fig. 1. *Involute profile, C_1 and C_2 rolling centrodes; I – wheel's rotation; II - rack gear's translation*

The involute of circle with radius R_b, see figure 1, is defined by vectorial equation

$$\overrightarrow{OM} = \overrightarrow{OT} + \overrightarrow{TM}, \qquad (1)$$

where:

$$\overrightarrow{OT} = -R_b\cos(\alpha+\varphi)\cdot\vec{i} + R_b\sin(\alpha+\varphi)\cdot\vec{j}; \qquad (2)$$

$$\begin{aligned}\overrightarrow{TM} = &-R_b\cdot\varphi\sin(\alpha+\delta+\varphi)\vec{i} + \\ &+R_b\cos(\alpha+\delta+\varphi)\vec{j}\end{aligned}, \qquad (3)$$

with φ variable parameter, and α and δ constant geometrical values.

In this way, the involute equations, the Euler equations, are defined as:

$$E\begin{vmatrix}X = -R_b\cos(\alpha+\varphi)- \\ -R_b\sin(\alpha+\delta+\varphi)\sin(\alpha+\varphi); \\ Y = R_b\sin(\alpha+\varphi)- \\ -R_b\sin(\alpha+\delta+\varphi)\cos(\alpha+\varphi).\end{vmatrix} \qquad (4)$$

The two constants, δ and α are deduced from equations:

$$\begin{vmatrix}x^2+y^2=R_b^2; \\ x^2+y^2=R_r^2,\end{vmatrix} \qquad (5)$$

$R_b = R_r\cos\alpha$, (α normalized, current α=20°) and R_r radius of C_1centrode.

The rolling process kinematics for the two centrodes includes the movements:

- the rotation of C_1centrode and, joined with this, the *XYZ* reference system around the *z* axis of the global reference system,

$$x = \omega_3^T(\theta)\cdot X, \qquad (6)$$

with θ rotation angular parameter;

- the translation of the C_2centrode, along the η axis:

$$x = \xi + a;\ a = \begin{pmatrix}-R_r \\ -R_r\cdot\theta\end{pmatrix}. \qquad (7)$$

In this way, the relative motion of the *XYZ* reference system regarding the $\xi\eta\zeta$ system, from (6) and (7), results in form:

$$\xi = \omega_3^T(\theta)\cdot X + \begin{pmatrix}R_r \\ R_r\cdot\theta\end{pmatrix}. \qquad (8)$$

From (8) results:

$$\begin{aligned}\xi = &\left[-R_b\cos(\alpha+\varphi)-\right. \\ &\left.-R_b(\alpha+\delta+\varphi)\sin(\alpha+\varphi)\right]\cos\theta- \\ &-\left[R_b\sin(\alpha+\varphi)-\right. \\ &\left.-R_b(\alpha+\delta+\varphi)\cos(\alpha+\varphi)\right]\sin\theta+R_r; \\ \eta = &\left[-R_b\cos(\alpha+\varphi)-\right. \\ &\left.-R_b(\alpha+\delta+\varphi)\sin(\alpha+\varphi)\right]\sin\theta+ \\ &+\left[R_b\sin(\alpha+\varphi)-\right. \\ &\left.-R_b(\alpha+\delta+\varphi)\cos(\alpha+\varphi)\right]\cos\theta+R_r\theta.\end{aligned} \qquad (9)$$

The involutes family, in the $\xi\eta\zeta$ reference system, can be restrained in form:

$$(E)_\theta\begin{vmatrix}\xi = -R_b\cos(\alpha+\varphi-\theta)- \\ -R_b(\alpha+\delta+\varphi)\sin(\alpha+\varphi-\theta)+R_r; \\ \eta = R_b\sin(\alpha+\varphi-\theta)- \\ -R_b(\alpha+\delta+\varphi)\cos(\alpha+\varphi-\theta)+R_r\cdot\varphi.\end{vmatrix} \qquad (10)$$

The enwrapping of involutes family (10) represents the profile of rack gear in $\xi\eta$ plane.

3. RACK GEAR RECIPROCALLY ENWRAPPING WITH INVOLUTES' FAMILY

It is proposed the determination of enwrapping condition by the method of "generating trajectories family" [9].

For this, are defined the directrix parameters of the normal to the involute (the $\overrightarrow{TM}$ versor, see figure 1) denoted with $\vec{n}_E$:

$$\vec{n}_E = -\sin(\alpha+\varphi)\vec{i} - \cos(\alpha+\varphi)\vec{j}. \qquad (11)$$

The direction of normal to the involute (4) may be written as:

$$(N_E)\begin{vmatrix}X = -R_b\cos(\alpha+\varphi)-R_b(\alpha+\delta+\varphi)\cdot \\ \cdot\sin(\alpha+\varphi)-\lambda_N\sin(\alpha+\varphi); \\ Y = R_b\sin(\alpha+\varphi)-R_b(\alpha+\delta+\varphi)\cdot \\ \cdot\cos(\alpha+\varphi)-\lambda_N\cos(\alpha+\varphi),\end{vmatrix} \qquad (12)$$

with λ_N variable scalar value.

It is possible to define the trajectories family for the normal (12) in the generating movement (8), by transforming:

$$(N_E)_\theta : \xi = \omega_3^T(\theta)\cdot N_E - a\,, \qquad (13)$$

or, developed:

$$(N_E)_\theta \left| \begin{aligned} &\xi = -R_b\cos(\alpha+\varphi-\theta) - \\ &-R_b(\alpha+\delta+\varphi)\cdot\sin(\alpha+\varphi-\theta) - \\ &-\lambda_N\sin(\alpha+\varphi-\theta)+R_r; \\ &\eta = R_b\sin(\alpha+\varphi-\theta) - \\ &-R_b(\alpha+\delta+\varphi)\cdot\cos(\alpha+\varphi-\theta) - \\ &-\lambda_N\cos(\alpha+\varphi-\theta)+R_r\cdot\theta. \end{aligned}\right. \qquad (14)$$

In the centrodes’ rolling process, the family of normals to the involute profile (14) must pass through the gearing pole:

$$P\left| \begin{aligned} &\xi = 0; \\ &\eta = R_r\cdot\theta. \end{aligned}\right. \qquad (15)$$

From the requirement that the equations (14) accomplish the conditions (15), it is obtained an equations system, where is eliminated the λ parameter:

$$\lambda_N = \frac{-R_b\cos(\alpha+\varphi-\theta)}{\sin(\alpha+\varphi-\theta)} - \frac{R_b(\alpha+\delta+\varphi)\sin(\alpha+\varphi-\theta)+R_r}{\sin(\alpha+\varphi-\theta)};$$

$$\textit{and} \qquad (16)$$

$$\lambda_N = \frac{R_b\sin(\alpha+\varphi-\theta)}{\cos(\alpha+\varphi-\theta)} - \frac{R_b(\alpha+\delta+\varphi)\cos(\alpha+\varphi-\theta)+R_r\theta-R_r\theta}{\cos(\alpha+\varphi-\theta)}.$$

From the equality of the two conditions for λ_N results the specifically enwrapping condition:

$$\varphi = \theta\,. \qquad (17)$$

In this way, joining the equations which represent the involute family (10) with the enwrapping condition (17) results the parametrical equations of the rack gear’s profile *S*:

$$S\left| \begin{aligned} &\xi = -R_b\cos\alpha - R_b(\alpha+\delta+\varphi)\sin\alpha + R_r; \\ &\eta = R_b\sin\alpha - R_b(\alpha+\delta+\varphi)\cos\alpha + R_r\cdot\theta. \end{aligned}\right. \qquad (18)$$

For the involute profile of circle with R_b radius the relations are known:

$$\delta = \tan\alpha - \alpha \quad \textit{and} \quad R_b = R_r\cdot\cos\alpha\,, \qquad (19)$$

so, the equations (18) can be brought to form:

$$S\left| \begin{aligned} &\xi = -R_b\cdot\varphi\cdot\sin\alpha\cdot\cos\alpha; \\ &\eta = R_b\cdot\varphi\cdot\sin\alpha\cdot\sin\alpha, \end{aligned}\right. \qquad (20)$$

which, with notation:

$$u = R_b\cdot\varphi\cdot\sin\alpha\,, \qquad (21)$$

may be described by the following equations:

$$S\left| \begin{aligned} &\xi = -u\cdot\cos\alpha; \\ &\eta = u\cdot\sin\alpha, \end{aligned}\right. \qquad (22)$$

representing the profile of the generating rack gear in the plane $\xi\eta$.

If we accept that the involute flank is a cylindrical surface, with frontal profile (4) and with a generatrix parallel with the *Z* axis, as is the case of straight teeth wheel, having the generatrix equation

$$Z = t\,,\ (t\text{ — variable}), \qquad (23)$$

then, the generating rack gear’s equations are:

$$S\left| \begin{aligned} &\xi = -u\cdot\cos\alpha; \\ &\eta = u\cdot\sin\alpha; \\ &\zeta = t, \end{aligned}\right. \qquad (24)$$

with *u* and *t* variables parameters, see figure 2.

The equations (24) represent, in principle, a cylindrical surface with generatrix parallels with the axis ζ. The “cylindrical” surface is reduced, in this case, to a plane parallel with the axis ζ.

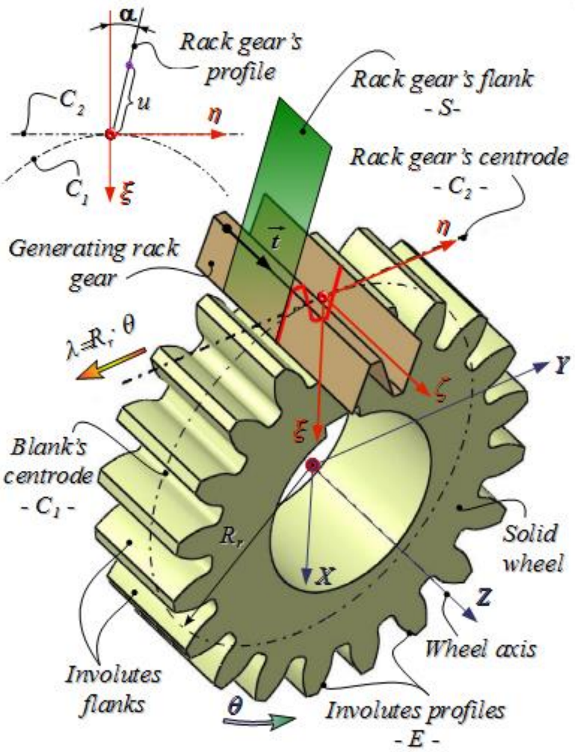

Fig. 2. *The rack gear’s flank — the S plane, reciprocally enveloping with the involute flanks; reference systems; generating kinematics*

4. ANALITICAL PROFILING OF HOB MILL

The hob mill, reciprocally enwrapping with flanks or the involute teethed wheel, accepts as primary peripheral surface a cylindrical helical surface with constant pitch, which admits a contact point (characteristic point) with the involute cylindrical flank in the frontal plane of the teethed wheel, constituting a pair or enveloping surfaces.

In the following is proposed the determination of conjugated worm (the primary peripheral surface of hob mill) using the complementary method of the "generating relative trajectories".

The method is applied in analytical form, using the principle of intermediary surface (generating rack gear) in linear contact with the involute cylindrical flank of the teethed wheel — the characteristic curve between the wheel and the rack. Also, the helical surface of the hob mill is profiled as surface reciprocally enveloping with the generating rack gear's flank.

The two characteristic curves are simultaneous onto the rack gear's surface. The instantaneous intersection point of these curves represents the characteristic point — the point where the hob mill generates the involute flank of the teethed wheel.

In figure 3, are presented the reference system associated with the blank, the generating rack gear and the future hob mill.

$\xi_1\eta_1\zeta_1$ — relative reference system joined with the $\vec{A}$ axis, which results from the decomposition of the helical movement (the worm's helix with axis $\vec{V}$ and *p* helical parameter);

$X_1Y_1Z_1$ — relative reference system joined with the worm representing the primary peripheral surface of the worm, enwrapping of the involute flank of the wheel;

It is considered that the generating movement of the worm — the primary peripheral surface of the future hob mill ($\vec{V}$, *p*) is decomposed in two movements:

- translation along the generatrix of the rack gear's surface, with direction $\vec{t}_0$ parallel with the blank's axis — the axis *Z*;

- rotation around the $\vec{A}$ axis parallel with the $\vec{V}$ axis of the hob mill and at the distance a_0 from this (see figure 3):

$$a_0 = p \cdot \tan\theta . \qquad (25)$$

So, the decomposition of the helical movement

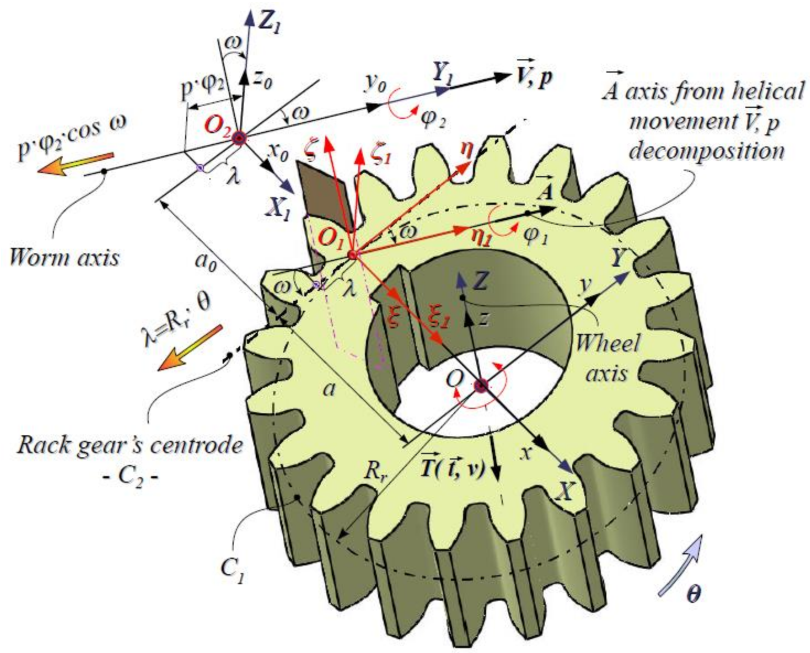

Fig. 3. *Reference systems — generating kinematics*

They are defined:

xyz is the global reference system with *z* axis overlapped by the teethed wheel;

$x_0y_0z_0$ — auxiliary reference system joined with the axis of the future hob mill (y_0 the axis of the worm);

XYZ — relative reference system joined with the involute flank of the teethed wheel (initially overlapped to the *xyz* reference system);

$\xi\eta\zeta$ — relative reference system, joined with the centrode associated with the rack gear, with axis parallels to *xyz*; the η axis is overlapped by the C_2centrode of the rack gear.

may be symbolically represented:

$$\left(\vec{V}, p\right) \sim T\left(\vec{t}, v\right) + \left(\vec{A}, \omega_A\right). \qquad (26)$$

with *v* and ω_A - movement parameters.

The flank of the generating worm results as enwrapping of the generating rack gear's surface (23) in the helical motion ($\vec{V}$, *p*) or in the movements assembly in which it is decomposed — rotation with axis $\vec{A}$ and translation along the generatrix of $\vec{t}$ versor, see figure 2.

We have to notice that, in the motion $\vec{T}(\vec{t},v)$, the generating rack gear is self-generated. So, the characteristic curve of the surface (24) will not depend on this component of movement, but only on the rotation around the $\vec{A}$ axis. This approach reduces the calculation effort, making easy to determine the characteristic curve between the rack's surface and the worm which constitutes the primary peripheral surface of hob mill.

The generating process kinematics includes:

- the translation of the rack gear:

$$x = \xi + a;\ a = \begin{pmatrix} -R_r \\ -\lambda \\ 0 \end{pmatrix}; \qquad (27)$$

- the rotation of worm around the $\vec{V}$ axis:

$$x_0 = \omega_2^T(\varphi_2)\cdot X_1; \qquad (28)$$

- the relative position of fixed reference systems:

$$x_0 = \beta\cdot(x - a_0);\ a_0 = \begin{pmatrix} -R_r - a_0 \\ 0 \\ 0 \end{pmatrix}; \qquad (29)$$

- the relative motion between the mobile reference systems (the relative motion of the rack gear regarding the reference system joined with hob mill):

$$X_1 = \omega_2(\varphi_2)\cdot\beta\cdot(\xi + a - a_0), \qquad (30)$$

where:

$$\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega \\ 0 & \sin\omega & \cos\omega \end{pmatrix}, \qquad (31)$$

$$\omega_2(\varphi_2) = \begin{pmatrix} \cos\varphi_2 & 0 & -\sin\varphi_2 \\ 0 & 1 & 0 \\ \sin\varphi_2 & 0 & \cos\varphi_2 \end{pmatrix}, \qquad (32)$$

and

$\lambda = p\cdot\varphi_2\cdot\cos\omega$, (see (28) and fig. 3), (33)

ω — angle between $\vec{V}$ axis and the frontal plane of the involute teethed wheel.

In this way, the matrix form of the surface generated by the rack gear flank is deduced regarding the hob mill's reference system:

$$\begin{pmatrix} X_1 \\ Y_1 \\ Z_1 \end{pmatrix} = \begin{pmatrix} \cos\varphi_2 & 0 & -\sin\varphi_2 \\ 0 & 1 & 0 \\ \sin\varphi_2 & 0 & \cos\varphi_2 \end{pmatrix}\cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega \\ 0 & \sin\omega & \cos\omega \end{pmatrix}\cdot \left[\begin{pmatrix} -u\cdot\cos\alpha \\ u\cdot\sin\alpha \\ t \end{pmatrix} + \begin{pmatrix} a_0 \\ -p\cdot\varphi_2\cdot\cos\omega \\ 0 \end{pmatrix}\right]. \qquad (34)$$

After developments, results the parametrical equations of the rack gear's movement regarding the $\vec{V}$ axis (helical movement with *p* helical parameter):

$$(S)_{\varphi_2}\left|\begin{aligned} &X_1 = [-u\cos\alpha + a_0]\cos\varphi_2 - \\ &-t\sin\varphi_2\cos\omega - \\ &-[u\sin\alpha - p\varphi_2\cos\omega]\sin\omega\sin\varphi_2; \\ &Y_1 = [-p\varphi_2\cos\omega + u\sin\alpha] - t\sin\omega; \\ &Z_1 = [-u\cos\alpha + a_0]\sin\varphi_2 + \\ &+t\cos\varphi_2\cos\omega + \\ &+[u\sin\alpha - p\varphi_2\cos\omega]\cos\omega\cos\varphi_2. \end{aligned}\right. \qquad (35)$$

We have to notice that the translation of the $\xi\eta\zeta$ reference system (joined with the rack gear) is defined in correlation with the translation along the $\vec{V}$ axis in the helical motion:

$$\lambda = p\cdot\varphi_2\cdot\cos\omega. \qquad (36)$$

The enwrapping of the surfaces' family $(S)_{\varphi_2}$ regarding the helical surface's reference system represents the primary peripheral surface of the future hob mill.

The enwrapping of the $(S)_{\varphi_2}$ family, (35), is determined based on the complementary theorem of the generating trajectories [9].

The versor of the normal to the *S* surface is calculated:

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\cos\alpha & \sin\alpha & 0 \\ 0 & 0 & 1 \end{vmatrix} = \sin\alpha\cdot\vec{i} + \cos\alpha\cdot\vec{j} \qquad (37)$$

and, in this way, the direction of the normal to *S*, in the current point, can be determined, (24) and (37):

$$\vec{N}_S\left|\begin{aligned} &\xi = -u\cdot\cos\alpha + k\cdot\sin\alpha; \\ &\eta = u\cdot\sin\alpha + k\cdot\cos\alpha; \\ &\zeta = t, \end{aligned}\right. \qquad (38)$$

where *k* is a variable scalar value.

The $\vec{A}$ axis is parallel with $\vec{V}$ axis and at distance a_0 from this (a_0 is measured along the $x \equiv x_0$ axis), see figure 3.

The $\vec{N}_S$ normal equations, (38), are written in the $\xi_1\eta_1\zeta_1$ reference system joined with $\vec{A}$ and with axis parallels to $X_1Y_1Z_1$ by transformations:

$$\begin{pmatrix}\xi_1\\ \eta_1\\ \zeta_1\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\omega & -\sin\omega\\ 0 & \sin\omega & \cos\omega\end{pmatrix}\cdot\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix}. \tag{39}$$

From (38) and (39) results the form of the $\vec{N}_S$ normal, in the $\xi_1\eta_1\zeta_1$ reference system with η_1 axis overlapped to $\vec{A}$ axis (see figure 3):

$$\left(\vec{N}_S\right)_{\xi_1\eta_1\zeta_1}\left|\begin{aligned}&\xi_1=-u\cdot\cos\alpha+k\cdot\sin\alpha;\\ &\eta_1=\left[u\cdot\sin\alpha+k\cdot\cos\alpha\right]\cdot\cos\omega-\\ &-t\cdot\sin\omega;\\ &\zeta_1=\left[u\cdot\sin\alpha+k\cdot\cos\alpha\right]\cdot\sin\omega+\\ &+t\cdot\cos\omega.\end{aligned}\right. \tag{40}$$

By rotating the $\left(\vec{N}_S\right)_{\xi_1\eta_1\zeta_1}$ normal, expressed in the $\xi_1\eta_1\zeta_1$ reference system, around the $\vec{A}$ axis, with angle φ_1:

$$\begin{pmatrix}\xi_1\\ \eta_1\\ \zeta_1\end{pmatrix}=\begin{pmatrix}\cos\varphi_1 & 0 & -\sin\varphi_1\\ 0 & 1 & 0\\ \sin\varphi_1 & 0 & \cos\varphi_1\end{pmatrix}\cdot \begin{pmatrix}-u\cdot\cos\alpha+k\cdot\sin\alpha\\ \left(u\cdot\sin\alpha+k\cdot\cos\alpha\right)\cdot\cos\omega-t\cdot\sin\omega\\ \left(u\cdot\sin\alpha+k\cdot\cos\alpha\right)\cdot\sin\omega+t\cdot\cos\omega\end{pmatrix}, \tag{41}$$

is determined the normal's family at the rack gear's flank, in the $\xi_1\eta_1\zeta_1$ reference system:

$$\left(N_S\right)_{\varphi_1}\left|\begin{aligned}&\xi_1=\left(-u\cdot\cos\alpha+k\cdot\sin\alpha\right)\cdot\cos\varphi_1-\\ &-\left[\left(u\cdot\sin\alpha+k\cdot\cos\alpha\right)\cdot\sin\omega+\right.\\ &\left.+t\cdot\cos\omega\right]\cdot\sin\varphi_1;\\ &\eta_1=\left(u\cdot\sin\alpha+k\cdot\cos\alpha\right)\cdot\cos\omega-\\ &-t\cdot\sin\omega;\\ &\zeta_1=\left(-u\cdot\cos\alpha+k\cdot\sin\alpha\right)\cdot\sin\varphi_1+\\ &+\left[\left(u\cdot\sin\alpha+k\cdot\cos\alpha\right)\cdot\sin\omega+\right.\\ &\left.+t\cdot\cos\omega\right]\cdot\cos\varphi_1.\end{aligned}\right. \tag{42}$$

According to the complementary theorem of the generating trajectories [9], is imposed the condition that the normals' family $\left(N_S\right)_{\varphi_1}$ intersect the $\vec{A}$ axis, which, in the $\xi_1\eta_1\zeta_1$ reference system has equations:

$$A\left|\begin{aligned}&\xi_1=0;\\ &\zeta_1=0.\end{aligned}\right. \tag{43}$$

If it is eliminated the k parameter from (42) and (43) equations assembly, the specific enwrapping condition is determined. By equalizing the ξ_1 coordinate from (42) and (43), results:

$$k=\frac{-u\cos\alpha\cos\varphi_1-u\sin\alpha\sin\omega\sin\varphi_1}{-\sin\alpha\cos\varphi_1+\cos\alpha\sin\omega\sin\varphi_1}-\\ -\frac{t\cos\omega\sin\varphi_1}{-\sin\alpha\cos\varphi_1+\cos\alpha\sin\omega\sin\varphi_1} \tag{44}$$

and, similarly, for ζ_1:

$$k=\frac{-u\cos\alpha\sin\varphi_1+u\sin\alpha\sin\omega\cos\varphi_1}{-\sin\alpha\sin\varphi_1-\cos\alpha\sin\omega\cos\varphi_1}-\\ -\frac{t\cos\omega\cos\varphi_1}{-\sin\alpha\sin\varphi_1-\cos\alpha\sin\omega\cos\varphi_1}. \tag{45}$$

By equalizing the equations (44) and (45), results the condition

$$t=-u\cdot\frac{\tan\omega}{\sin\alpha}, \tag{46}$$

representing the enwrapping specific condition, which, associated with rack gear's flanks family, in the relative motion regarding the $X_1Y_1Z_1$ reference system (35) determines the primary peripheral surface of the future hob mill:

$$\Sigma\left|\begin{aligned}&X_1=\left(-u\cos\alpha+a_0\right)\cos\varphi_2+\\ &+\frac{u\tan\omega}{\sin\alpha}\sin\varphi_2\cos\omega-\\ &-\left(u\sin\alpha-p\varphi_2\cos\omega\right)\sin\omega\sin\varphi_2;\\ &Y_1=\left(-p\varphi_2\cos\omega+u\sin\alpha\right)\cos\omega+\\ &+\frac{u\tan\omega}{\sin\alpha}\sin\omega;\\ &Z_1=\left(-u\cos\alpha+a_0\right)\sin\varphi_2+\\ &+\left(u\sin\alpha-p\varphi_2\right)\sin\omega\cos\varphi_2-\\ &-\frac{u\tan\omega}{\sin\alpha}\cos\omega\cos\varphi_2.\end{aligned}\right. \tag{47}$$

The equations (47) represent a cylindrical helical surface with Y_1 axis and p helical parameter.

5. HOB MILL'S HELICAL SURFACE FORM

The worm's form is analyzed as helical surface reciprocally enwrapping with a curl of cylindrical surfaces with involute profile in frontal plane (straight teethed wheel). From (47), for

$$\varphi_2=0, \tag{48}$$

the generatrix of the helical surface is determined:

$$G\begin{vmatrix} X_1 = -u \cdot \cos\alpha + a; \\ Y_1 = u \cdot \sin\alpha \cdot \cos\omega + u \cdot \dfrac{\tan\omega}{\sin\alpha} \cdot \sin\omega; \\ Z_1 = u \cdot \sin\alpha \cdot \sin\omega - u \cdot \dfrac{\tan\omega}{\sin\alpha} \cdot \cos\omega, \end{vmatrix} \quad (49)$$

representing a straight line for which the directrix parameters can be defined:

$$G\begin{vmatrix} l_1 = -\cos\alpha; \\ m_1 = \sin\alpha \cdot \cos\omega + \dfrac{\sin^2\omega}{\sin\alpha\cos\omega}; \\ n_1 = \sin\alpha\sin\omega - \dfrac{\sin\omega}{\sin\alpha}. \end{vmatrix} \quad (50)$$

At the same time, the directrix parameters of the $\vec{V} = \vec{j}$ axis are defined in the $X_1Y_1Z_1$ reference system:

$$V\,|\,l_2 = 0;\ m_2 = 1;\ n_2 = 0. \quad (51)$$

The relative position of the two straight lines $\vec{V}$ and $\vec{G}$ is examined. The minimum distance between the two lines can be calculated in vectorial form, see figure 4.

The angle between the directions $\vec{G}$ and $\vec{V}$ is defined as:

$$\tan\Phi = \frac{\left|\vec{G} \times \vec{V}\right|}{\vec{G} \cdot \vec{V}}. \quad (52)$$

The vectorial product $\vec{G} \times \vec{V}$ is calculated:

$$\vec{G} \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\cos\alpha & \left(\sin\alpha\cos\omega + \dfrac{\sin^2\omega}{\sin\alpha\cos\omega}\right) & \left(\sin\alpha\sin\omega - \dfrac{\sin\omega}{\sin\alpha}\right) \\ 0 & 1 & 0 \end{vmatrix} \quad (53)$$

or

$$\vec{G} \times \vec{V} = -\left(\sin\alpha\sin\omega - \frac{\sin\omega}{\sin\alpha}\right) \cdot \vec{i} - \cos\alpha \cdot \vec{k} \quad (54).$$

The modulus of the $\vec{G} \times \vec{V}$ vectorial product is:

$$\left|\vec{G} \times \vec{V}\right| = \sqrt{\cos^2\alpha + \left(\sin\alpha\sin\omega - \frac{\sin\omega}{\sin\alpha}\right)^2}. \quad (55)$$

Also, is calculated the scalar product

$$\vec{G} \cdot \vec{V} = \left(\sin\alpha\cos\omega + \frac{\sin^2\omega}{\sin\alpha\cos\omega}\right). \quad (56)$$

In this way, from (52) and the definitions of the vectorial product modulus (55) and scalar product (56), results the definition of the angle between *G* and *V*:

$$\tan\Phi = \frac{\sqrt{\cos^2\alpha + \left(\sin\alpha\sin\omega - \dfrac{\sin\omega}{\sin\alpha}\right)^2}}{\sin\alpha\cos\omega + \dfrac{\sin^2\omega}{\sin\alpha\cos\omega}}. \quad (57)$$

The distance between the two straight lines is defined by equation:

$$d = \frac{\Delta}{\vec{G} \cdot \vec{V}}, \quad (58)$$

where Δ is the mixed product $\left(\vec{r}, \vec{G}, \vec{V}\right)$ or

$$\Delta = \begin{vmatrix} a_0 & 0 & 0 \\ -\cos\alpha & \left(\sin\alpha\cos\omega + \dfrac{\sin^2\omega}{\sin\alpha\cos\omega}\right) & \left(\sin\alpha\sin\omega - \dfrac{\sin\omega}{\sin\alpha}\right) \\ 0 & 1 & 0 \end{vmatrix} \quad (59)$$

$$\Delta = a_0\left(\sin\alpha\sin\omega - \frac{\sin\omega}{\sin\alpha}\right) \quad (60)$$

and $\vec{r}$ is the vector which link two points belonging to the lines $\vec{G}$ and $\vec{V}$, see figure 4:

$$\vec{r} = a_0 \cdot \vec{i}. \quad (61)$$

So, the minimum distance between the two lines, $\vec{G}$ and $\vec{V}$, is

$$d = \frac{a_0\left(\sin\alpha\sin\omega - \dfrac{\sin\omega}{\sin\alpha}\right)}{\sin\alpha\cos\omega + \dfrac{\sin^2\omega}{\sin\alpha\cos\omega}}. \quad (62)$$

We have to notice that the minimum distance (62) represents the radius of a cylinder which admits $\vec{V}$ as axis and also admits as tangent direction the Σhelical surface's straight line generatrix, (47), see figure 4.

The expressions (57) and (62) can be processed, see Appendix 1, so result, from (57),

$$\tan\Phi = \frac{\cos\omega}{\sqrt{\tan^2\alpha + \sin^2\omega}} . \qquad (63)$$

Also, from (62), see Appendix 2 and figure 5, results

$$d = \frac{p\cdot\cos\omega}{\sqrt{\tan^2\alpha + \sin^2\omega}} . \qquad (64)$$

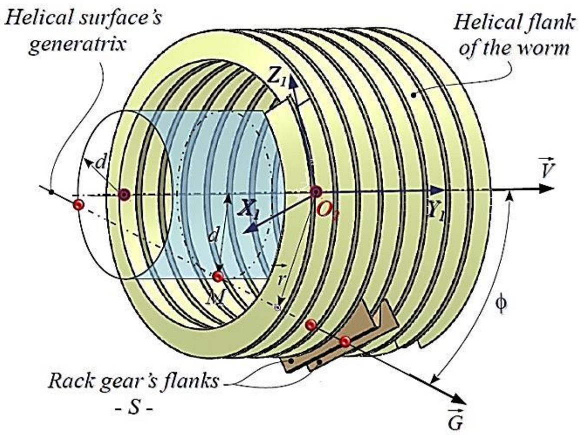

Fig. 4. *The G and V lines, the minimum distance d and the ϕ angle*

From (64), results

$$\frac{d}{p} = \frac{\cos\omega}{\sqrt{\tan^2\alpha + \sin^2\omega}} = \tan\theta , \qquad (65)$$

see figure 5 and Appendix 2.

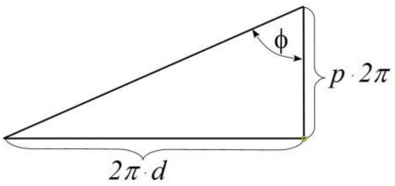

Fig. 5. *The unfold of helix with p parameter on the cylinder with radius d*

It is obviously that

$$\theta = \Phi , \qquad (66)$$

so, the G line is tangent to the worm helix on the cylinder with radius d and axis $\vec{V}$.

So, the worm conjugated with the involute teethed wheel is an involute worm.

6. CONCLUSIONS

It is proved, by the complementary method of the generating trajectories, that the conjugated worm of a straight involute teethed wheel is a cylindrical helical surface with constant pitch — an involute worm.

The involute worm — ruled worm, allows a straight line generatrix tangent to the helix onto the cylinder with radius R_{rt} of the hob mill. The application demonstrates that the worm's generatrix is on the distance d, see (62), from the axis of hob mill and at angle Φ from this axis. The angle Φ represents the angle of the helix onto the cylinder with radius R_{rs}, condition which represents the definition of the involute worm.

As a consequence, the only helical surface reciprocally enwrapping with an involute teethed wheel is an involute worm accepted as peripheral primary surface of a generating hob mill.

A rigorous hob mill may be constituted only if the cutting edges of the teeth are curves which belong to the involute worm.

APPENDIX 1

Expressions of the Φ angle and d “minimum distance”

*

$$\sqrt{\cos^2\alpha+\left(\sin\alpha\cdot\sin\omega-\frac{\sin\omega}{\sin\alpha}\right)^2}=\sqrt{\cos^2\alpha+\frac{\cos^4\alpha}{\sin^2\alpha}\cdot\sin^2\omega}=\sqrt{\cos^2\alpha+\left(1+\frac{\cos^2\alpha}{\sin^2\alpha}\right)\cdot\sin^2\omega}=$$
$$=\sqrt{\cos^2\alpha+\left(1+\frac{1}{\tan^2\alpha}\right)^2\cdot\sin^2\alpha}=\frac{\cos\alpha}{\tan\alpha}\sqrt{\tan^2\alpha+\sin^2\alpha}. \quad (67)$$

**

$$a_0\cdot\left(\sin\alpha\cdot\sin\omega-\frac{\sin\omega}{\sin\alpha}\right)=\frac{a_0\cdot\left(\sin^2\alpha-1\right)}{\sin\alpha}\cdot\sin\omega=\frac{a_0}{\sin\alpha}\cdot\cos^2\alpha\sin\omega=\frac{a_0\cdot\cos\alpha\cdot\sin\omega}{\tan\alpha}. \quad (68)$$

***

$$\tan\Phi=\frac{\cos\alpha\sqrt{\tan^2\alpha+\sin^2\omega}}{\tan\alpha\left(\sin\alpha\cos\omega+\dfrac{\sin^2\omega}{\sin\alpha\cos\omega}\right)}=\frac{\sqrt{\tan^2\alpha+\sin^2\omega}}{\dfrac{\tan\alpha}{\cos\alpha}\left(\dfrac{\sin^2\alpha\cos^2\omega+\sin^2\omega}{\sin\alpha\cos\omega}\right)}=$$
$$=\frac{\sqrt{\tan^2\alpha+\sin^2\omega}}{\dfrac{1}{\cos\omega}\left(\tan^2\alpha\cos^2\omega+\dfrac{\sin^2\omega}{\cos^2\alpha}\right)}=\frac{\sqrt{\tan^2\alpha+\sin^2\omega}}{\dfrac{1}{\cos\omega}\left[\tan^2\alpha\left(1-\sin^2\omega\right)+\dfrac{\sin^2\omega}{\cos^2\alpha}\right]}= \quad (69)$$
$$=\frac{\cos\omega\sqrt{\tan^2\alpha+\sin^2\omega}}{\tan^2\alpha-\tan^2\alpha\sin^2\omega+\dfrac{\sin^2\omega}{\cos^2\alpha}}=\frac{\cos\omega\sqrt{\tan^2\alpha+\sin^2\omega}}{\tan^2\alpha-\sin^2\omega+\left(\dfrac{1-\sin^2\alpha}{\cos^2\omega}\right)}.$$

So:

$$\tan\Phi=\frac{\cos\omega\sqrt{\tan^2\alpha+\sin^2\omega}}{\tan^2\omega+\sin^2\omega}=\frac{\cos\omega}{\sqrt{\tan^2\alpha+\sin^2\omega}}. \quad (70)$$

APPENDIX 2

$$\tan V=\frac{d}{p}. \quad (71)$$

$$d=\frac{\Delta}{\vec{G}\cdot\vec{V}}=\frac{a_0\cdot\left(\sin\alpha\cdot\sin\omega-\dfrac{\sin\omega}{\sin\alpha}\right)}{\sin\alpha\cdot\cos\omega+\dfrac{\sin^2\omega}{\sin\alpha\cdot\cos\omega}}=\frac{\dfrac{a_0\cdot\cos\alpha\cdot\sin\omega}{\tan\alpha}}{\sin\alpha\cdot\cos\omega+\dfrac{\sin^2\omega}{\sin\alpha\cdot\cos\omega}}. \quad (72)$$

$$d=\frac{a_0\cdot\sin\omega}{\sqrt{\tan^2\alpha+\sin^2\omega}}=\frac{p\cdot\cos\omega}{\sqrt{\tan^2\alpha+\sin^2\omega}}. \quad (73)$$

$$\tan\theta=\frac{d}{p}=\frac{a_0\cdot\sin\omega}{p\cdot\sqrt{\tan^2\alpha+\sin^2\omega}}=\frac{p\cdot\cos\omega}{p\cdot\sqrt{\tan^2\alpha+\sin^2\omega}}=\frac{\cos\omega}{\sqrt{\tan^2\alpha+\sin^2\omega}}. \quad (74)$$

It is obviously that the *Φ* angle between the $\vec{G}$ and $\vec{V}$ vectors and the *V* angle — the angle of helix onto the cylinder with radius *d* of the worm — are equals, see figure 5.

APPENDIX 3

Geometrical values of the teeth and the dimensions of the hob mill

- circular pitch of the teeth — straight teethed wheel

$$p_c = \frac{2 \cdot \pi \cdot R_r}{z_d}; \tag{75}$$

z_d is the number of teeth of the wheel;
- normal pitch of the generating rack gear,

$$p_{n_{rack}} = p_c = \frac{2 \cdot \pi \cdot R_r}{z_t}; \tag{76}$$

- angle of hob mill's helix onto the cylinder with radius R_{rs}:

$$\tan\omega = \frac{p_{ax\_worm}}{2 \cdot \pi \cdot R_{rt}} \tag{77}$$

where p_{ax_worm} is the axial pitch of the hob mill and R_{rt} is the rolling radius of the hob mill;
- $a_0 = p \cdot \tan\theta$ — the relation (26) may be rewritten as:

$$\omega = \frac{\pi}{2} - \theta, \tag{78}$$

where θ is the angle between the generatrix of the helical surface and the axis of hob mill;
- from equality condition between the normal pitch of the rack gear and the hob mill:

$$\frac{2 \cdot \pi \cdot R_r}{z_t} = 2 \cdot \pi \cdot R_{rt} \cdot \tan\omega \cdot \cos\omega, \tag{79}$$

results

$$\sin\omega = \frac{R_{rp}}{R_{rt}} \cdot \frac{1}{z_t}; \tag{80}$$

- the helical parameter of the hob mill

$$p = \frac{p_{ax\_worm}}{2 \cdot \pi} = \frac{R_{rp}}{z_t} \cdot \frac{1}{\cos\omega} \cdot \frac{1}{2 \cdot \pi}; \tag{81}$$

- the a_0 distance

$$a_0 = \frac{R_{rp}}{z_t} \cdot \frac{1}{\sin\omega_s} = \frac{R_{rp}}{z_t} \cdot \frac{R_{rt}}{R_{rp}} \cdot z_t = R_{rt}. \tag{82}$$

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