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STRESS STATE IN AN ELASTIC BENT PLANE WITH TWO IDENTICAL CIRCULAR HOLES

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ABSTRACT

The paper presents a comparison between the stress states obtained in an elastic bent plane with two identical circular holes. The plane is bent in a manner that the neutral axis of the plane is normal to the axis formed by the two centres of the holes, in the middle point. For the analytical solution the elastic potential is obtained as a sum between the elastic potential of the compact plane and an auxiliary potential, having a form given by Jeffery. The boundary conditions were imposed to the resulting potential. The plane is loaded with a linear distributed force. For the analysis with finite elements, the load was applied using two concentrated couples. The effect of the two holes is strictly locally and does not depend on the loading mode, reflecting the Saint-Venant principle. The fact is confirmed by the perfect agreement between the plots of the stresses, obtained by the analytical and numerical methods. The principal shearing plots are especially useful as they may be obtained experimentally as well, by photoelastic method, namely isochromatics curves. Finally, theoretically and numerically isochromatics are compared to the experimental ones obtained by Mesmer, cited by Frocht.

KEYWORDS: elastic plane, bending, stress concentration factor

1. INTRODUCTION

The aim of the paper is finding the stress state in an elastic plane with two identical circular holes. The elastic plane is subjected to pure bending and the neutral axis is also a symmetry axis for the two holes, as shown in Fig. 1. The stress state is found both by means of analytical and numerical methods and the results are compared with experimental data given in literature.

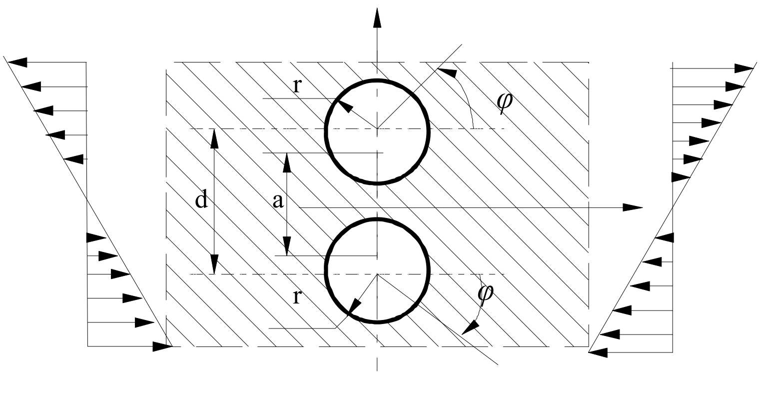


Fig. 1. Geometry and loading

2. THEORETICAL REMARKS

The analytical solution of the problem is found using bipolar co-ordinates, (α,β), correlated with Cartesian co-ordinates, (x,y), by the following relations:

$$x = a\frac{sin(\beta)}{cosh(\alpha)-cos(\beta)},\qquad y = a\frac{sinh(\alpha)}{cosh(\alpha)-cos(\beta)} \tag{1}$$

In bipolar co-ordinated, written under the form (1), the level curves $\alpha = const.$ and $\beta = const.$ are difficult to recognise. The equation (1) can be written in complex form as it follows:

$$\alpha + i\beta = ln\frac{x+i(y+a)}{x+i(y-a)} = ln\frac{z+ia}{z-ia}, \tag{2}$$

where a point from the Cartesian plane *Oxy* is characterised by affix $z = x+iy$. Another form for mapping (1) , due to Spiegel, [1], is:

$$x^2 + \left[y-\frac{a}{th(\alpha)}\right]^2 = \left[\frac{a}{sh(\alpha)}\right]^2; \qquad \left[x-\frac{a}{tan(\beta)}\right]^2 + y^2 = \left[\frac{a}{sin(\beta)}\right]^2. \tag{3}$$

The first relation from equations (3) represents a circle with the centre on *Oy* axis in the point *(0,a/th(α)),* with radius *a/sh(α)*, and the second relation is a circle with the centre on *Ox* axis, in the point (*a/tan(β),0*), having the radius *a/sin(β)*. Both of the level curves families are plotted in Fig. 2 and it can be seen that all the circles of the second family, ($\beta = const.$), have the same radical axis, that is they all pass through the points $A_1(0,a)$ and $A_2(0,-a)$.

The elastic compact plane bended at infinity has Airy’s function characteristic, in Cartesian co-ordinates, $U(x,y)$. This function is necessary for constructing the Airy’s function for the considered plane, with holes. The form of the Airy’s function is:

$$U(x,y) = \frac{1}{6}my^3 \tag{4}$$

where *m* is a constant with measured in *N/m*. The potential (4) is written in bipolar co-ordinates and divided by *J*, as Jeffery shows, [2], where *J* has the form:

$$J = \frac{a}{cosh(\alpha)-cos(\beta)}, \tag{5}$$

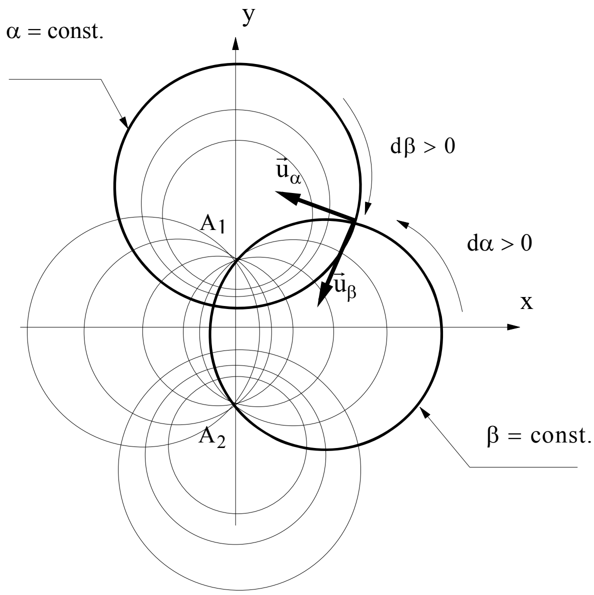


Fig. 2. Level curves of bipolar co-ordinates

The boundary conditions are:

$$\sigma_a(\alpha,\beta)\big|_{a=\pm\alpha_0} = 0; \tag{6.a}$$

$$\tau_{a\beta}(\alpha,\beta)\big|_{a=\pm\alpha_0} = 0, \tag{6.b}$$

where $\alpha = \pm acosh(d/2r)$ represents the contour of the holes in bipolar co-ordinates.

The potential for the compact elastic plane in bipolar co-ordinates is:

$$U(\alpha,\beta) = \frac{1}{6}ma^2\frac{sinh(\alpha)^3}{[cosh(\alpha)-cos(\beta)]^2} \tag{7}$$

In order to impose the above boundary conditions, Jeffery, [2], demonstrates that the Airy’s function, $V(\alpha,\beta)$, is obtained adding an auxiliary potential $\Phi(\alpha,\beta)$ to the potential of the compact plane, $U(\alpha,\beta)$. The auxiliary potential, proposed by Jeffery, has the expression:

$$\begin{aligned}\Phi(\alpha,\beta) &= \{B_0\alpha + K\,ln[ch(\alpha)-cos(\beta)]\}[ch(\alpha)-cos(\beta)] + \\ &[A_1ch(2\alpha)+B_1+C_1sh(2a)]\,cos(\beta)+[A'_1\,ch(2\alpha)+C'_1\,sh(2a)]\,sin(\beta) \\ &+\sum_{k=2}^{\infty}\left\{\begin{array}{l}[A_kch[(k+1)\alpha]+B_kch[(k-1)\alpha]+C_ksh[(k-1)\alpha]+D_ksh[(k-1)\alpha]]\,cos(k\beta)+ \\ +[A'_k\,ch[(k+1)\alpha]+B'_k\,ch[(k-1)\alpha]+C'_k\,sh[(k-1)\alpha]+D'_k\,sh[(k-1)\alpha]]\,sin(k\beta)\end{array}\right\},\end{aligned} \tag{8}$$

The coefficients of the potential $\Phi(\alpha,\beta)$ are determined imposing both boundary conditions and the continuity conditions at infinity (as the domain is boundless) for stresses. The last condition can be mathematically expressed under the form:

$$\lim_{\substack{\alpha\to 0\\ \beta\to 0}}[\Phi(\alpha,\beta)] = 0\ , \tag{9}$$

The condition (9) is equivalent to the relation:

$$\sum_{k=1}^{n}(A_k + B_k) = 0 \tag{10}$$

With the potential $V(\alpha,\beta)$ found, the stresses can be determined in bipolar co-ordinates, using the relations:

$$\sigma_a = \frac{1}{a}\left\{[cosh(\alpha)-cos(\beta)]\frac{\partial^2}{\partial\beta^2} - sinh(\alpha)\frac{\partial}{\partial\alpha} - sin(\beta)\frac{\partial}{\partial\beta} + cosh(\alpha)\right\}V,$$

$$\sigma_\beta = \frac{1}{a}\left\{[cosh(\alpha)-cos(\beta)]\frac{\partial^2}{\partial\alpha^2} - sinh(\alpha)\frac{\partial}{\partial\alpha} - sin(\beta)\frac{\partial}{\partial\beta} + cos(\beta)\right\}V, \tag{11}$$

$$\tau_{a\beta} = -\frac{1}{a}[cosh(\alpha)-cos(\beta)]\frac{\partial^2 V}{\partial\alpha\partial\beta}.$$

A powerful method for validation of the analytical results is the photoelastic method. It consists in comparing the theoretical and experimental isochromatics (defined as the curves where $\tau_{max} = const.$) and isoclinics (the curves where the principal stresses have the same direction).

The equations for the isoclinics in Cartesian co-ordinate, given by Frocht, [3], are:

$$tan(2\phi) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \tag{12}$$

In the present paper only the qualitative aspects are envisaged, by comparing analytical, numerical and experimental isochromatics patterns.

The isochromatics have the following relation:

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} = const. \tag{13}$$

The numerical procedure allows finding the Cartesian stresses. For a quantitative comparison between numerical and analytical results, the relations of the Cartesian stresses are needed. The change of the stress tensor components from bipolar to Cartesian co-ordinates is:

$$\sigma_x = \frac{sinh(\alpha)^2 sin(\beta)^2\sigma_a + [cosh(\alpha)cos(\beta)-1]^2\sigma_\beta - \left\{[cosh(\alpha)cos(\beta)-1]^2 - sinh(\alpha)^2 sin(\beta)^2\right\}\tau_{a\beta}}{[cosh(\alpha)-cos(\beta)]^2},$$

$$\sigma_y = \frac{[cosh(\alpha)cos(\beta)-1]^2\sigma_a + sinh(\alpha)^2 sin(\beta)^2\sigma_\beta + \left\{[cosh(\alpha)cos(\beta)-1]^2 - sinh(\alpha)^2 sin(\beta)^2\right\}\tau_{a\beta}}{[cosh(\alpha)-cos(\beta)]^2},. \tag{14}$$

$$\tau_{xy} = \frac{sinh(\alpha)sin(\beta)[cosh(\alpha)cos(\beta)-1](\sigma_a - \sigma_\beta) + [cosh(\alpha)cos(\beta)-1]^2\sigma_\beta \ldots + \left\{[cosh(\alpha)cos(\beta)-1]^2 - sinh(\alpha)^2 sin(\beta)^2\right\}\tau_{a\beta}}{[cosh(\alpha)-cos(\beta)]^2}$$

3. RESULTS

The analytical isochromatics obtained in the present paper are given in Fig. 3a, and compared to the experimental isochromatics, given by Mesmer and quoted by Frocht, [3], as shown in Fig. 3b.

The stress concentrator effect of the holes is pointed out in Fig. 4 and Fig. 5, where the hoop stress variation on the contour of the holes is plotted.

Both evaluations show a very good agreement between numerical and analytical (exact) results.

The bending of an elastic part with two holes was analysed using finite element method, using CATIA.

The stress concentration factor can not be determined using the classic definition, as the normal stress due to bending varies, depending on the position of the fibre. In order to appreciate the concentrator effect of the holes, the maximum hoop stress was divided by the stress for the compact plane, from the point where the centre of the hole would be, under the same loading, was represented. The results are shown in Fig. 6. Due to hyperbolic functions in the relations of the stresses, the calculus was made only for a ratio distance/radius smaller than 10. A very interesting issue is the tendency of asymptotic variation of this dimensionless stress, towards a value around 3. This observation may be used in simplifying the calculus of the maximum hoop stress, meaning that a rapid estimation of maximum hoop stress can be find by only multiplying the stress form the compact plane corresponding to the centre of the hole with 3, and the the evaluation is good enough for the holed plane.

In Fig. 7 the numerical results obtained are compared with the isochromatics found analytically, in Mathcad. In Fig. 8, the normal maximum stress patterns are presented, both for FEA method, Fig. 8a, and for analytically method Fig. 8b. A very good agreement between the two methods can be emphasised. Slight differences occur in regions far away from the neutral axis, due to different initial modelling assumptions of the elastic body: modelling: with a finite part in CATIA, (the edge effect can be remarked) and modelling with an infinite plane in Mathcad.

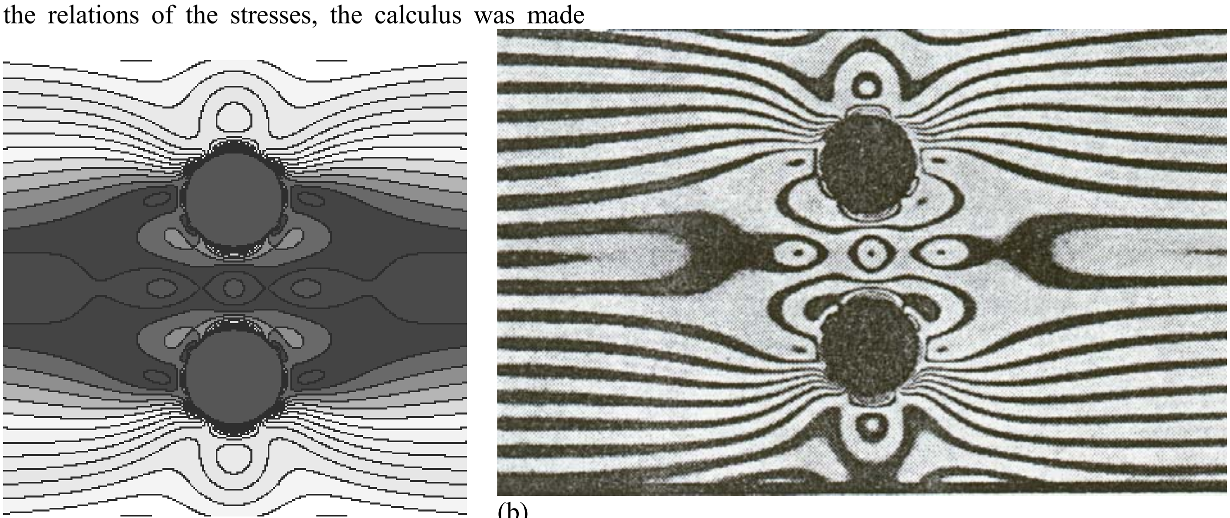


Fig. 3. Theoretical, (a), and experimental, [3], (b) isochromatics patterns

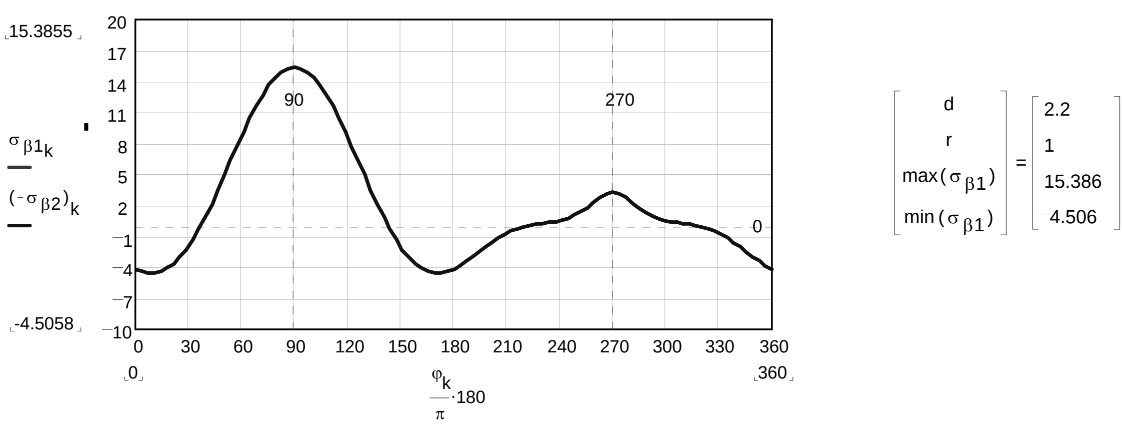
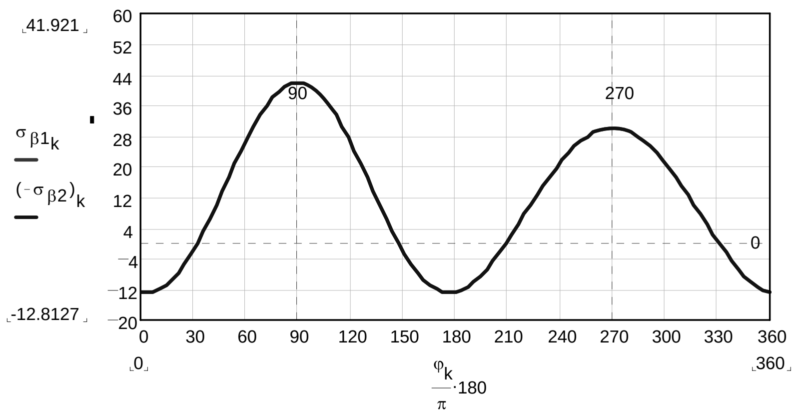
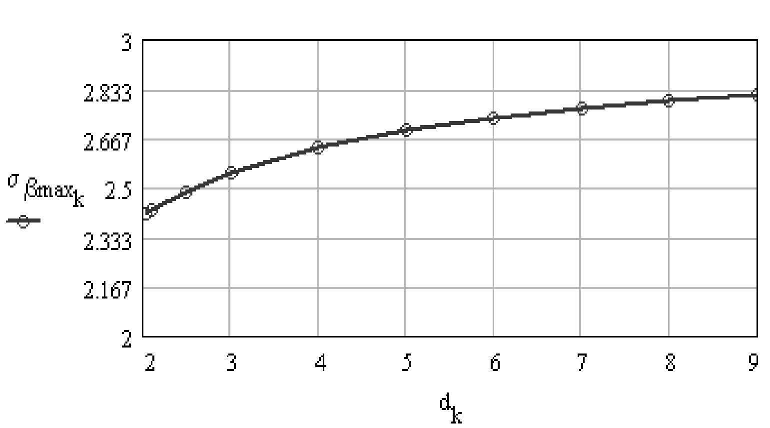


Fig. 4. Hoop stress variation on the contour of the holes (case of nearby holes)



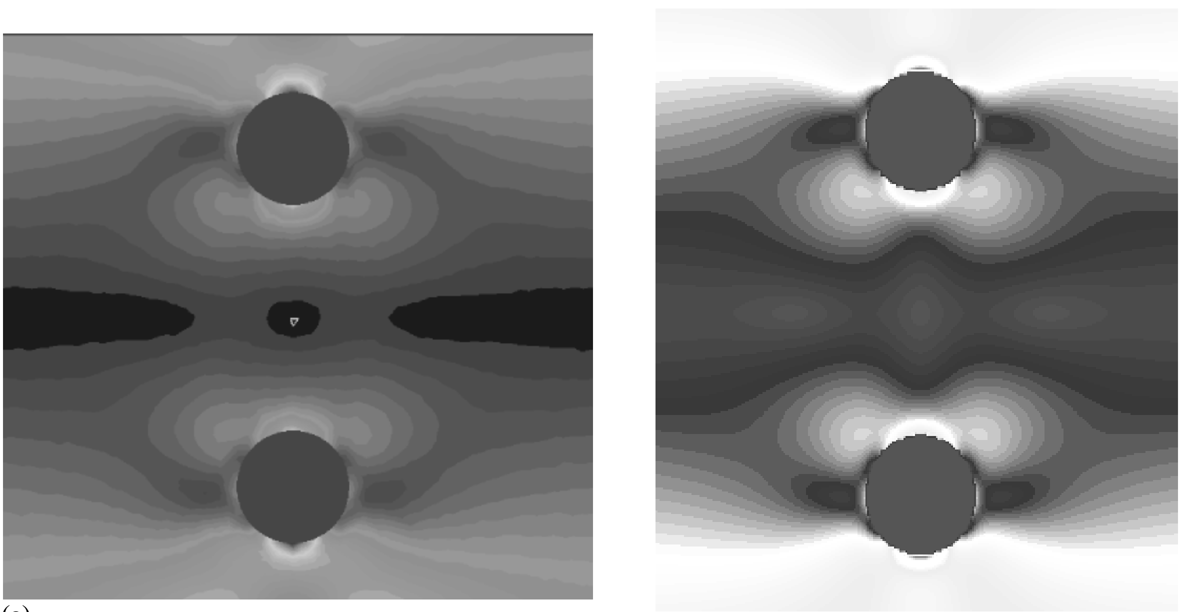
$$\begin{bmatrix} d \\ r \\ \max(\sigma_{\beta 1}) \\ \min(\sigma_{\beta 1}) \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 41.921 \\ -12.813 \end{bmatrix}$$

Fig. 5. Hoop stress variation on the contour of the holes (case of distant holes)



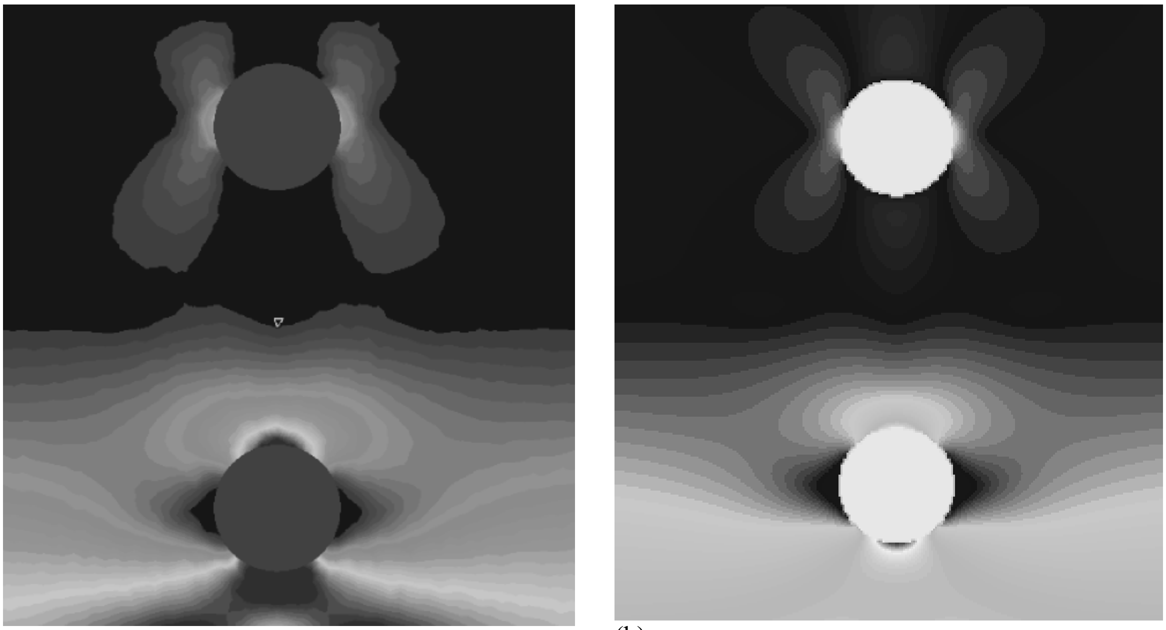
$$d = \begin{bmatrix} 2.05 \\ 2.1 \\ 2.5 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{bmatrix} \quad \sigma_{\beta max} = \begin{bmatrix} 2.417 \\ 2.425 \\ 2.488 \\ 2.551 \\ 2.638 \\ 2.697 \\ 2.739 \\ 2.769 \\ 2.795 \\ 2.814 \end{bmatrix}$$

Fig. 6. Stress concentration factor on the contour of the hole - variation with distance between the two holes



(a) (b)

Fig. 7. Isochromatic patterns obtained with finite element analysis (a) and analytically (b)



(a) (b)

Fig. 8. Normal maximum stresses obtained with finite element analysis (a) and analytically (b)

4. CONCLUSIONS

The main conclusions that can be drawn from the present work are:

- Stress concentration factor of the holes is higher than the case when the holes centres are placed on neutral axis, [5].
- The hoop stress on the contour of the holes in points symmetrically places with respect to the neutral axis, has the same modulus but with opposite signum.
- The maximum hoop stress appears on the contour of the holes, in the bent half-plane and in the most remote point from the neutral axis;
- A very good agreement between theoretically, numerically and experimentally obtained results is attained, drawing the conclusion that numerical methods can provide correct solutions for a certain problem where analytical methods are ineffective.
- It was found the tendency of asymptotic variation of the dimensionless stress, towards a value around 3.
- This observation may be used in simplifying the calculus of the maximum hoop stress: a rapid estimation of maximum hoop stress can be found by only multiplying the stress form the compact plane corresponding to the centre of the hole with 3, and the evaluation is good enough for the holed plane.

REFERENCES

[1] **Spiegel, M. R.**, *Theory and Problems of Vector Analysis and an Introduction to Tensor Analysis*, Schaum Publishing & Co. New York, 1959.

[2] **Jeffery, G. B.,** *Plane Stress and Plane Strain in Bipolar Coordinates*, Philos. Trans. Of the Royal Society, London, A221, 1921, p. 265-293.

[3] **Frocht, M. M.,** *Photoelasticity*, New York, John Wiley & Sons, Vol. I, 1941, Vol. 2, 1948

[4] **Dundurs J, Markenscoff X.** *The Sternberg-Koiter conclusion and other anomalies of the concentrated couple* [J], *.Journal of Applied Mechanics*, 1989, **56**(2):p. 240–245.

[5] **Alaci, S.** , Stress State in a Bent Plane with two Different Circular Holes (the axis of Centers is Parallel to the Neutral Axis of the Plane), Annals of the Oradea University, fasc. Management and technological Engineering, Vol V (XV), on CD, p. 84 in Extended abstracts, Oradea, 2006.